Magnetic Tension and the Geometry of the Universe

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The vector nature of magnetic fields and the geometrical interpretation of gravity introduced by general relativity guarantee a special coupling between magnetism and spacetime curvature. This magneto-geometrical interaction effectively transfers the tension properties of the field into the spacetime fabric, triggering a variety of effects with profound implications. Given the ubiquity of magnetic fields in the universe, these effects could prove critical. We discuss the nature of the magnetic-field–curvature coupling and illustrate some of its potential implications for cosmology.

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Despite the widespread presence of magnetic fields in the universe [1], studies of their potential cosmological implications remain relatively underdeveloped. It has long been thought, however, that magnetic fields might have played a role during the formation and the evolution of the observed large scale structure [2]. Recently, this idea has received renewed interest manifested by the increasing number of related papers that have appeared in the literature [3,4]. Nevertheless, there are still only a few fully relativistic approaches available. Most treatments are either Newtonian or semirelativistic. As such, they are bound to exclude certain features of the magnetic nature. Two key features are the vector nature of the field and the tension properties of magnetic force lines. In general relativity vector fields have quite a different status than ordinary scalar sources, such as the energy density and pressure of matter. The geometrical nature of Einstein's theory guarantees that vectors are directly coupled to the spacetime curvature. This special interaction is manifested in the Ricci identity

$$2\nabla_{[a}\nabla_{b]}B_c = R_{abcd}B^d, \qquad (1)$$

applied here to the magnetic vector B_a, where R_{abcd} is the spacetime Riemann tensor. The Ricci identity plays a fundamental role in the mathematical formulation of general relativity. Essentially, it is the definition of spacetime curvature itself. The Ricci identity also leads to a direct coupling between magnetism and spatial geometry. Indeed, projecting Eq. (1) into the instantaneous rest space of a comoving observer we arrive at

$$2D_{[a}D_{b]}B_c = -2\omega_{ab}h_c^d\dot{B}_d + \mathcal{R}_{dcba}B^d. \qquad (2)$$

In the above D_a is the projected covariant derivative operator, ω_{ab} is the vorticity tensor, and h_{ab}, $\mathcal{R}_{abcd}$ are, respectively, the metric and the Riemann tensor of the observer's rest space. Note that the overdot indicates differentiation along the observer's world line. The vorticity term appears because generally the observer's motion is not hypersurface orthogonal. For our purposes, however, the key quantity is the last one on the right-hand side of Eq. (2). Its presence illustrates the direct coupling between magnetic fields and spatial geometry. We call this special interaction the *magnetocurvature coupling*. This coupling goes beyond the standard interplay between matter and geometry as introduced by the Einstein field equations. In fact, it makes the magnetic field an inseparable part of the spacetime fabric by effectively transferring its properties to the spacetime itself. The key property appears to be the tension of the magnetic lines of force.

Magnetic fields transmit stresses between regions of material particles and fluids. The field exerts an isotropic pressure in all directions and carries a tension along the magnetic lines of force. Each small flux tube behaves like an infinitely elastic rubber band, while neighboring tubes expand against each other under their own pressure. Equilibrium exists only when a balance between pressure and tension is possible. To unravel these tension properties, let us consider a pure magnetic field. Its energy-momentum tensor decomposes as

$$T_{ab} = \tfrac{1}{2}B^2u_au_b + \tfrac{1}{6}B^2h_{ab} + \Pi_{ab}, \qquad (3)$$

where $B^2 = B_aB^a$ and $\Pi_{ab} = (B^2/3)h_{ab} - B_aB_b$. Thus, the field behaves as an imperfect fluid with energy density $\rho_{\rm m} = B^2/3$, isotropic pressure $p_{\rm m} = B^2/6$, and anisotropic pressure Π_{ab}. Note that, in the absence of electric fields, the electromagnetic Poynting vector is zero and Eq. (3) contains no energy-flux vector. The tension properties of the field are incorporated in the symmetric trace-free tensor Π_{ab}. They emerge when we take the eigenvalues of Π_{ab} orthogonal and along the direction of the magnetic force lines. Orthogonal to B_a one finds two positive eigenvalues equal $1/3$ each. Thus, the magnetic pressure perpendicular to the field lines is positive, reflecting their tendency to push each other apart. In the B_a direction, however, the associated eigenvalue is $-2/3$ and the magnetic pressure is negative. The minus sign reflects the tension properties of the field lines and their tendency to remain as "straight" as possible.

The magnetic effects on the fluid propagate through Euler's formula. For a barotropic, infinitely conductive, magnetized medium the nonlinear Euler equation is [4]

$$(\rho + p + \tfrac{2}{3}B^2)A_a = -c_s^2 D_a\rho - \varepsilon_{abc}B^b \,\mathrm{curl}B^c - A^b\Pi_{ba}\,, \qquad (4)$$

where A_a is the fluid 4-acceleration $c_s^2 = \dot{p}/\dot{\rho}$ is the sound speed squared, and ε_{abc} is the spatial alternating tensor. The projected gradient of Eq. (4) facilitates a more detailed study of the magnetic behavior. In particular, it separates the effects of the field's tension from those of the ordinary (i.e., the positive) magnetic pressure. To illustrate how and also to reveal the role of the spacetime curvature, we consider a weakly magnetized Bianchi I background. Its generic anisotropy will help us to identify the magnetic tension contributions easier. Employing Eq. (2) we arrive at the linear expression [5]

$$\rho(1 + w)D_bA_a = -c_s^2 D_bD_a\rho - \frac{c_s^2}{\rho(1+w)}B_aB^cD_bD_c\rho + B^cD_cD_bB_a - \tfrac{1}{2}D_bD_aB^2 - \tfrac{4}{3}\Theta B_aB^c\omega_{bc} + \mathcal{R}_{acbd}B^cB^d, \qquad (5)$$

with $w = p/\rho$. Let us concentrate on the four magnetic terms at the end of Eq. (5). The first two also appear in Newtonian studies, whereas the last two are the relativistic corrections. Since $D^aB_a = 0 = B^aB^b\omega_{ab}$, the first standard term and the first of the relativistic corrections are irrelevant for our purposes. Here we focus upon the magnetocurvature term at the end of Eq. (5), where the spatial curvature tensor is twice contracted along the field lines. The directional dependence on B_a ensures that $\mathcal{R}_{acbd}B^cB^d$ also conveys the magnetic tension effects. This is also implied by the sign difference between the magnetogeometrical term and the standard gradient $D_bD_aB^2$, which carries the effects of the ordinary magnetic pressure. Note that the magnetocurvature stress in Eq. (5) is always normal to the field lines, as the symmetries of the Riemann tensor confirm. Also, it closely resembles the classical curvature stress exerted by distorted magnetic field lines (see, e.g., [6]). This resemblance becomes more apparent when we linearize (5) about a Friedmann-Robertson-Walker (FRW) background [see Eqs. (7) and (8)]. The difference is that, in the relativistic case, the distortion of the field pattern is triggered by the spacetime geometry itself. In fact, the magnetocurvature term in Eq. (5) effectively injects the tension properties of the field into the spacetime fabric. The implications are widespread and far from trivial.

Consider a general spacetime filled with a magnetized, highly conductive, perfect fluid. Its volume expansion is governed by the nonlinear Raychaudhuri equation [4]

$$\tfrac{1}{3}\Theta^2 q = \tfrac{1}{2}(\rho + 3p + B^2) + 2(\sigma^2 - \omega^2) - \nabla^a A_a - \Lambda\,, \qquad (6)$$

where q is the deceleration parameter, σ^2 and ω^2 are the shear and vorticity magnitudes, respectively, and Λ is the cosmological constant. The state of the expansion is determined by the sign of the right-hand side of Eq. (6). Positive terms decelerate the universe while negative ones lead to acceleration. Clearly, conventional matter and shear effects slow the expansion down. On the other hand, vorticity and a positive cosmological constant accelerate the universe. Hence, every term on the right-hand side of Eq. (6) has a clear kinematical role with the exception of ∇^aA_a. The latter can be either positive or negative, depending on the specific form of the 4-acceleration. In our case A_a obeys the nonlinear Euler formula given by Eq. (4). In a weakly magnetized, slightly inhomogeneous and anisotropic, almost-FRW universe Eqs. (4) and (6) linearize to give [7]

$$\tfrac{1}{3}\Theta^2 q = \tfrac{1}{2}\rho(1 + 3w) + \frac{c_s^2\Delta}{(1+w)a^2} + \frac{c_a^2\mathcal{B}}{2(1+w)a^2} - \frac{2kc_a^2}{(1+w)a^2} - \Lambda \qquad (7)$$

on using the trace of (5). In the above Δ and $\mathcal{B}$ describe scalar perturbations in the fluid and the magnetic energy densities, respectively, $c_a^2 = B^2/\rho$ is the square of the Alfvén speed, $k = 0, \pm 1$ is the background curvature index, and a is the scale factor. Given that in the linear regime the mean values of Δ and $\mathcal{B}$ are zero, one expects that on average Eq. (7) looks like

$$\tfrac{1}{3}\Theta^2 q = \tfrac{1}{2}\rho(1 + 3w) - \frac{2kc_a^2}{(1+w)a^2}\,, \qquad (8)$$

where $\Lambda = 0$ from now on. Note the magnetocurvature term on the right-hand side which results from the coupling between magnetism and geometry as manifested in Eq. (2). This term affects the expansion in two completely different ways depending on the sign of the background curvature. In particular, the magnetogeometrical effects slow the expansion down when $k = -1$ but tend to accelerate the expansion if $k = +1$. Such a behavior seems odd, especially since positive curvature is always associated with gravitational collapse. The explanation lies in the elastic properties of the field lines. As curvature distorts the magnetic force lines their tension backreacts giving rise to a restoring magnetocurvature stress [5]. The magnetic backreaction has kinematical, dynamical, as well as geometrical implications. In Eq. (8), for example, the tension of the field adjusts the expansion rate of the universe to minimize the kinematical effects of curvature. As a result the expansion rate is brought closer to that of a flat FRW model. Overall, it looks as though the elastic properties of the field have been transferred into space. According to Eq. (8), the magnetocurvature effects also depend on the material component of the universe. When dealing with conventional matter (i.e., for $0 \leq w \leq 1$) the most intriguing cases occur in positively curved spaces [7]. In particular, when $w = 1$ (i.e., for stiff matter) the Alfvén speed grows as $c_a^2 \propto a^2$ and the magnetocurvature term in Eq. (8) becomes time independent. In this case the field acts as an effective positive cosmological constant. For radiation and dust, on the other hand, $c_a^2 = \mathrm{const}$ and $c_a^2 \propto a^{-1}$, respectively. In these cases the magnetocurvature term is no longer time independent but drops with time mimicking a time-decaying quintessence. The coupling between magnetism and geometry also means that even weak magnetic fields can have a strong impact if the curvature is strong. To demonstrate how this might happen, consider a weakly magnetized spatially open cosmology filled with nonconventional matter (i.e., $k = -1$ and $-1 \leq w < 0$). Scalar fields, for example, can have an effective equation of state that satisfies this requirement. Such models allow for an early curvature dominated regime with $\Omega \ll 1$. Given that $\rho \propto a^{-3(1+w)}$ and $c_{\rm a}^2 \propto a^{-1+3w}$, the magnetocurvature term in Eq. (8) can dominate the early expansion, even when the field is weak, if $-1 \leq w \leq -1/3$. In this case the accelerated inflationary phase, which otherwise would have been inevitable, is suppressed. Instead of inflating the magnetized universe remains in a state of decelerated expansion. For $w = -1$, in particular, the mere presence of the field can inhibit the de Sitter inflationary regime if $\Omega < 0.5$ [7]. This result has two implications. First, it challenges the widespread perception that magnetic fields are relatively unimportant for cosmology. Even weak fields can play a decisive role when the curvature is strong. Second, it casts doubt on the efficiency of standard inflation in the presence of primordial magnetism.

Let us now turn our attention to geometry and examine the implications of the magnetic tension for propagating gravitational radiation. To begin with, recall the tendency of the field lines to remain straight. If this property were transferred to the spacetime, we would expect to see a suppressing effect on gravity waves propagating through a magnetized region. Such damping should appear as a decrease in the wave's energy density and amplitude. To put this idea to the test we consider linear gravitational waves in a weakly magnetized almost-FRW universe, filled with a highly conductive medium. We also assume that the background spatial sections are flat and we address superhorizon scales only. Covariantly, gravity waves are described via the electric (E_{ab}) and the magnetic (H_{ab}) parts of the Weyl tensor [8]. Their magnitudes, $E^2 = E_{ab}E^{ab}/2$ and $H^2 = H_{ab}H^{ab}/2$, provide a measure of the wave's energy density and amplitude. Given that $H_{ab} = \mathrm{curl}\sigma_{ab}$, we can simplify the problem by replacing the magnetic Weyl tensor with the shear. Note that the field couples to gravitational radiation directly via the anisotropic magnetic stresses, which affect the propagation of both E_{ab} and σ_{ab} [9]. Having set the constraints that isolate tensor perturbations in a magnetized universe (see [9]), we arrive at the system

$$\begin{aligned} (E^2)^{\cdot} &= -2\Theta E^2 - \tfrac{1}{2}\rho(1+w)\mathcal{X} - \tfrac{1}{2}\Theta B^2\mathcal{E}\,, \\ (\sigma^2)^{\cdot} &= -\tfrac{4}{3}\Theta\sigma^2 - \mathcal{X} - \tfrac{1}{2}B^2\Sigma\,, \\ \dot{\mathcal{X}} &= -\tfrac{5}{3}\Theta\mathcal{X} - 2E^2 - \rho(1+w)\sigma^2 \\ &\quad - \tfrac{1}{2}B^2\mathcal{E} - \tfrac{1}{2}\Theta B^2\Sigma\,, \\ \dot{\mathcal{E}} &= -\Theta\mathcal{E} - \tfrac{1}{2}\rho(1+w)\Sigma - \tfrac{1}{3}\Theta B^2, \\ \dot{\Sigma} &= -\tfrac{2}{3}\Theta\Sigma - \mathcal{E} - \tfrac{1}{3}B^2, \end{aligned} \tag{9}$$

with $B^2 \propto a^{-4}$, $\mathcal{X} = E_{ab}\sigma^{ab}$, $\mathcal{E} = E_{ab}\eta^a\eta^b$, and $\Sigma = \sigma_{ab}\eta^a\eta^b$ ($\eta_a = B_a/\sqrt{B^2}$). The last two scalars are related via the Gauss-Codacci equation by

$$\mathcal{E} = \tfrac{1}{3}\Theta\Sigma + \tfrac{1}{3}B^2 + R\,, \tag{10}$$

where $R = [\mathcal{R}_{(ab)} - (\mathcal{R}/3)h_{ab}]\eta^a\eta^b$ describes spatial curvature distortions in the direction of the magnetic field lines. For radiation, the late-time solution for E^2 is [9]

$$\begin{aligned} E^2 &= \tfrac{4}{9}\left[E_0^2 + \frac{\sigma_0^2}{4t_0^2} - \frac{\mathcal{X}_0}{2t_0}\right]\left(\frac{t_0}{t}\right)^2 \\ &\quad - \tfrac{2}{9}\left(\tfrac{1}{6}B_0^2 + R_0\right)B_0^2\left(\frac{t_0}{t}\right)^2, \end{aligned} \tag{11}$$

with an analogous result for dust [9]. Note that the term in square brackets determines the magnetic-free case. According to Eq. (11), the field leaves the evolution rate of E^2 unchanged but modifies its magnitude. The magnetic impact is twofold. There is a pure magnetic effect, independent of the spatial curvature, which always suppresses the energy of the wave. It becomes apparent when we set $R_0 = 0$ in Eq. (11). This effect is the direct result of the magnetic tension. As the wave propagates it distorts the field lines which backreact by smoothing out any ripples in the spacetime fabric. The magnetically induced damping is proportional to the ratio B_0^2/E_0. Given the inherent weakness of gravitational radiation, the magnetic effects are potentially detectable even when relatively weak fields are involved. Solution (11) also reveals a magnetocurvature effect on gravitational radiation. This is encoded in the R_0 term and depends entirely on the spatial curvature. For $R_0 > 0$, namely, when the curvature distortion along the field lines is positive, the pure-magnetic damping is further enhanced. On the other hand, the suppressing effect of the field weakens if $R_0 < 0$. In fact, the field will increase the energy of the wave provided that $R_0 < -B_0^2/6$. These magnetogeometrical effects get stronger with increasing curvature distortion. Let us take a closer look at them. According to Eq. (10), the scalar R describes distortions in the local spatial curvature generated by the propagating magnetized gravity wave. Clearly, the magnetogeometrical term in Eq. (11) modifies the energy density of the wave in a way that always minimizes such curvature distortions. In other words, the pure magnetocurvature effect tends to preserve the spatial flatness of the background universe. Earlier, an analogous magnetocurvature effect was also observed on the expansion rate of spatially curved FRW universes. This pattern of behavior raises the question as to whether it reflects a generic feature of the magnetic nature. More specifically, one wonders if the tension properties of the magnetic force lines and the coupling between magnetism and spacetime curvature imply an inherent "preference" of the field for flat geometry. Let us take a more direct look at this possibility. Consider an almost-FRW magnetized universe and assume that the background spatial geometry is Euclidean. If $\mathcal{R}$ is the Ricci scalar of the perturbed spatial sections, then using the trace of Eq. (5) we obtain [4]

$$\dot{\mathcal{R}} = -\frac{2}{3}\left[1 + \frac{2c_{\mathrm{a}}^2}{3(1+w)}\right]\Theta\mathcal{R} + \frac{4c_{\mathrm{s}}^2\Theta}{3(1+w)a^2}\Delta + \frac{2c_{\mathrm{a}}^2\Theta}{3(1+w)a^2}\mathcal{B}\,. \qquad (12)$$

As expected, the expansion dilutes curvature distortions, caused by fluctuations in the fluid and the magnetic energy densities. Interestingly, the field also enhances the smoothing effect of the expansion. This effect results from the tension properties of the magnetic force lines, which tend to suppress curvature distortions. Given the weakness of the field (recall that $c_{\mathrm{a}}^2 \ll 1$), the magnetically induced smoothing is negligible compared to that caused directly by the expansion. Nevertheless, the tendency of the field to maintain the original flatness of the spatial sections is quite intriguing. It seems to support the idea that, given their tension properties and their direct coupling to curvature, magnetic fields might indeed have a natural preference for flat spaces.

The magnetocurvature effects presented here reveal a side of the magnetic nature which as yet remains unexplored. They derive from the vector nature of the field and from the geometrical approach to gravity adopted by general relativity. The latter allows a direct coupling between magnetism and curvature which effectively transfers the magnetic properties into space itself. The tension of the field lines appears to be the key property. Kinematically speaking, the magnetocurvature effects tend to accelerate spatially closed regions, while they decelerate those with open spatial curvature. Crucially, if the curvature input is strong, the overall impact can also be strong even when the field is weak. This challenges the widespread belief that, due to their perceived weakness, magnetic fields are relatively unimportant for cosmology. Inflationary scenarios allow for a strong-curvature regime during their very early stages. An initial curvature dominated epoch has never been considered a serious problem for inflation given the vast smoothing power of the accelerated expansion. It is during these early stages, however, that a weak magnetic presence is found capable of suppressing the accelerated phase in spatially open "inflationary" models. Such a result casts doubt on the efficiency, and potentially on the viability, of standard inflation in the presence of primeval magnetism. In fact, every cosmological model that allows for a strong-curvature regime and a weak magnetic field could be vulnerable to these magnetocurvature effects. The coupling between magnetism and spacetime curvature has also intriguing geometrical implications. It modifies the expansion of spatially closed, and open, FRW universes bringing the rate closer to that of a flat Friedmannian model. The tension of the field lines is found to suppress gravitational waves propagating through a magnetized region. Moreover, the combined magnetocurvature effects smooth out perturbations in the spatial curvature of a flat FRW universe and modulate the energy of gravity waves as if to preserve the background flatness. In short the magnetized space seems to react to curvature distortions showing, what one might interpret as, a preference for flat geometry. Given the ubiquity of magnetic fields in the universe, this unconventional behavior deserves further investigation as it could reflect a deeper interconnection between electromagnetism and geometry. This in turn could drastically change our views on the role of cosmic magnetism not only in cosmology but also in astrophysics. It is the aim of this Letter to bring these issues to light and draw attention to their potential implications.

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